Self-adjoint extensions of the curl operator

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Talk Abstract

In electromagnetism, for linear isotropic media the relation between the magnetic induction $\boldsymbol{B}$ and the magnetic field $\boldsymbol{H}$ is given by $\boldsymbol{B} = \mu\boldsymbol{H}$, the scalar function μ being the magnetic permeability. If displacement currents are neglected, as in the case of magnetostatic or eddy current problems, the current density $\boldsymbol{J}$ is given by $\boldsymbol{J} = \mathbf{curl}\,\boldsymbol{H}$. In this situation a magnetic field satisfying $\mathbf{curl}\,\boldsymbol{H} = \eta\boldsymbol{H}$, with η a scalar function, produces a vanishing magnetic force $\boldsymbol{J} \times \boldsymbol{B}$, and it is called a *force-free* field.

In fluid dynamics, a divergence-free field $\boldsymbol{u}$ satisfying $\mathbf{curl}\,\boldsymbol{u} = \eta\boldsymbol{u}$, with η a scalar function, is a steady solution of the Euler equations for incompressible inviscid flows (with pressure given by $p = -\frac{|\boldsymbol{u}|^2}{2}$), and it is called a *Beltrami* field.

Eigenfunctions of the curl operator are therefore force-free fields and Beltrami fields, and are of relevant physical interest. In particular, in plasma physics a magnetic field $\boldsymbol{H}$ which minimizes the magnetic energy with fixed helicity has to satisfy the equation $\mathbf{curl}\,\boldsymbol{H} = \lambda\boldsymbol{H}$ for some constant λ, thus it is an eigenfunction of the curl operator.

In this talk we are concerned with the formulation and analysis of the eigenvalue problem for the curl operator. Drawing inspiration from the results in [1] and extending the previous ones in [2], [3], [4], we prove that the curl operator is self-adjoint in L^2, provided that we choose a suitable domain of definition.

This domain is strictly larger than $H_0(\mathbf{curl};\Omega)$, the space of vector fields $\boldsymbol{v}$ belonging to $L^2(\Omega)$ together with $\mathbf{curl}\,\boldsymbol{v}$ and satisfying $\boldsymbol{v} \times \mathbf{n} = \mathbf{0}$ on the boundary $\partial\Omega$, and when Ω is topologically trivial is given by the space of vector fields for which $\mathbf{curl}\,\boldsymbol{v} \cdot \mathbf{n} = 0$ on the boundary. However, additional conditions must be imposed when Ω is not topologically trivial, and we show that a viable choice is imposing that the values of the line integrals of $\boldsymbol{v}$ on suitable homological cycles lying on the boundary are equal to 0.

Following the results in [5] we devise and analyze a saddle-point variational formulation for the spectral problem associated to the curl operator, and also briefly describe the guidelines of its numerical approximation by means of finite elements.

Keywords: curl operator, spectral problem, saddle-point variational formulation, finite element approximation.

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